*Int. J. Solids Structures* Vol. 15, pp. 885–897
Pergamon Press Ltd., 1979. Printed in Great Britain

# STEADY SOLUTIONS FOR A MOVING LOAD ON AN ELASTIC STRIP RESTING ON AN ELASTIC HALF PLANE†

G. G. ADAMS‡
Department of Mechanical Engineering, Northeastern University, Boston, MA 02115, U.S.A.

(*Received* 25 *October* 1978; *in revised from* 15 *March* 1979)

**Abstract**—An infinite elastic strip rests under gravity on a smooth elastic half plane. The strip is loaded by a steadily moving concentrated force which produces a partial separation of the layer from the foundation. Using the plane strain theory of elasticity, the resulting nonsymmetric mixed boundary value problem is reduced to singular integral equations over the unknown noncontact regions. For various material combinations and a range of force and speed, the location of the noncontact regions, the lower boundary displacements, and the foundation contact pressure are computed. Results for the corresponding problem with a stationary load are also given.

## INTRODUCTION

Contact problems between elastic strips and foundations have been the subject of many investigations. In[1] Keer *et al.* solved the plane and axisymmetric problems of an infinite elastic strip pressed against an elastic half plane in the absence of gravity. In[2] they generalized their treatment to include nonsymmetric load distributions and in[3] to deal with a layer with a slightly curved substrate. Ratwani and Erdogan solved the problem of a layer pressed against a half space by rigid stamps[4] and the axisymmetric problem with an elastic stamp[5]. The lifting of a semi-infinite strip lying on a rigid foundation is treated by Keer and Silva[6]. Civelik and Erdogan solved the problems of an infinite elastic strip resting on a rigid foundation and subjected to an upward directed load[7] and a downward directed load[8].

Such problems with steadily moving loads have just recently been investigated. The solution for a steadily moving upward load acting on Euler–Bernoulli and Timoshenko beams resting on a rigid foundation is given by Adams and Bogy[9] and by Adams[10] for an elastic strip. In[11] Adams solves the problem of a downward directed load acting on an elastic strip resting on a rigid foundation and in[12] for an elastic strip and half plane in the absence of gravity. The presence of gravity as well as a downward load generally produces two regions of separation which changes the method as well as the nature of the solution.

In this paper an elastic layer is pressed against a smooth elastic half plane by a steadily moving load as well as gravity. Solutions are obtained for an infinite strip and half plane which are then related through interface continuity conditions. Using integral transform techniques, the mixed boundary value problem is reduced to singular integral equations which are solved numerically by the method of Erdogan and Gupta[13]. Solutions are obtained for a range of speed and material combinations. These result in symmetric and nonsymmetric configurations for up to two noncontact regions. Finally the results obtained for the corresponding static problem are listed.

## PROBLEM FORMULATION

The problem under consideration deals with a two-dimensional (plane strain) homogeneous and isotropic elastic layer of constant thickness $h$ which is resting under gravity on a smooth elastic half plane of different material properties. The strip is subjected to a downward directed steadily moving concentrated force $P$, which tends to produce a partial separation of the layer

†This work was supported by the National Science Foundation under Grant ENG-77-22181.
‡Assistant Professor.

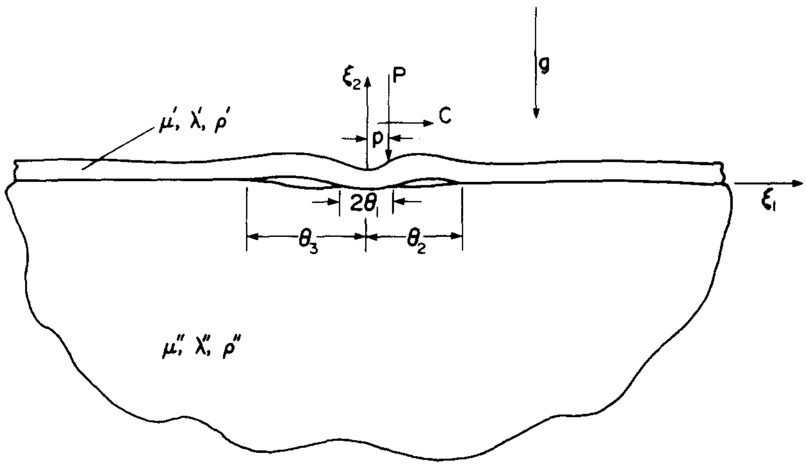


Fig. 1. An elastic layer resting on an elastic half-space ans subjected to a steadily moving load.

from the foundation (Fig. 1). The appropriate displacement equations of motion are

$$\mu \nabla^2 \bar{u}_\alpha + (\lambda + u) \frac{\partial v}{\partial x_\alpha} = \rho \frac{\partial^2 \bar{u}_\alpha}{\partial t^2} - F_\alpha,\ \alpha = 1,2 \tag{1}$$

where $\lambda$, $\mu$, $\rho$ are the Lame's constant, shear modulus, and mass density, $\bar{u}_\alpha$, $F_\alpha$ are the components of displacement, and body force, and where

$$v = \frac{\partial \bar{u}_1}{\partial x_1} + \frac{\partial \bar{u}_2}{\partial x_2},\ F_\alpha = (0, -\rho g).$$

By transferring to a dimensionless coordinate system moving with the load at constant speed, eqn (1) becomes

$$\begin{aligned} &(\delta^2 - \beta_2{}^2) \frac{\partial^2 u_1}{\partial \xi_1{}^2} + \frac{\partial^2 u_1}{\partial \xi_2{}^2} + (\delta^2 - 1) \frac{\partial^2 u_2}{\partial \xi_1 \partial \xi_2} = 0, \\ &\delta^2 \frac{\partial^2 u_2}{\partial \xi_2{}^2} + (1 - \beta_2{}^2) \frac{\partial^2 u_2}{\partial \xi_1{}^2} + (\delta^2 - 1) \frac{\partial^2 u_1}{\partial \xi_1 \partial \xi_2} = 0, \end{aligned} \tag{2}$$

where

$$\begin{gathered} \xi_1 - p = (x - ct)/h,\ \xi_2 = x_2/h, \\ u^*_\alpha(\xi_1, \xi_2) = (\mu/\rho g h^2) \bar{u}_\alpha(x_1, x_2, t) \\ u_\alpha = u^*_\alpha - u_\alpha{}^g,\ u_1{}^g = 0,\ u_2{}^g = -\xi_2(1 - \xi_2/2)/\delta^2 \\ \beta_\alpha = c/c_\alpha,\ \delta = c_1/c_2,\ c_1 = \sqrt{(\lambda + 2\mu)/\rho},\ c_2 = \sqrt{(\mu/\rho)}. \end{gathered} \tag{3}$$

Thus $u^*_\alpha$ is the dimensionless displacement, $u_\alpha{}^g$ is the displacement due to gravity, and the $u_\alpha$ in (2) is a residual displacement field. The stress-displacement relations are then written in the dimensionless form

$$\begin{gathered} \sigma_{\alpha\gamma} = \sigma^*_{\alpha\gamma} - \sigma^g_{\alpha\gamma} = (\delta^2 - 2) \left( \frac{\partial u_1}{\partial \xi_1} + \frac{\partial u_2}{\partial \xi_2} \right) \delta_{\alpha\gamma} + \frac{\partial u_\alpha}{\partial \xi_\gamma} + \frac{\partial u_\gamma}{\partial \xi_\alpha}, \\ \sigma^*_{\alpha\gamma} = \sigma_{\alpha\gamma}/\rho g h,\ \sigma^g_{11} = -(1 - \xi_2)(1 - 2/\delta^2),\ \sigma^g_{22} = -(1 - \xi_2),\ \sigma^g_{12} = 0. \end{gathered} \tag{4}$$

Now superscripts (′) and (″) will be used to refer to the strip and half plane respectively.

The boundary and continuity conditions to be applied are

$$\sigma'_{22}(\xi_1,\xi_2),\ \sigma'_{12}(\xi_1,\xi_2),\ \sigma'_{11}(\xi_1,\xi_2)\to 0 \text{ as } |\xi_1|\to\infty,\ 0<\xi_2<1 \tag{5}$$

$$\sigma''_{22}(\xi_1,\xi_2),\ \sigma''_{12}(\xi_1,\xi_2),\ \sigma''_{11}(\xi_1,\xi_2)\to 0 \text{ as } \xi_1^2+\xi_2^2\to\infty \tag{6}$$

$$\sigma'_{12}(\xi_1,1)=0,\ |\xi_1|<\infty \tag{7}$$

$$\sigma'_{22}(\xi_1,1)=-P\delta(\xi_1-p),\ |\xi_1|<\infty \tag{8}$$

$$\sigma'_{12}(\xi_1,0)=0,\ |\xi_1|<\infty \tag{9}$$

$$\sigma''_{12}(\xi_1,0)=0,\ |\xi_1|<\infty \tag{10}$$

$$\sigma'_{22}(\xi_1,0)=\delta_4\sigma''_{22}(\xi_1,0),\ |\xi_1|<\infty \tag{11}$$

$$u'_2(\xi_1,0)=\delta_1 u''_2(\xi_1,0),\ |\xi_1|<\theta_1,\ \xi_1>\theta_2,\ \xi_1<-\theta_3 \tag{12}$$

$$\sigma'_{22}(\xi_1,0)=1,\ \theta_1<\xi_1<\theta_2,\ -\theta_3<\xi_1<-\theta_1 \tag{13}$$

where $\delta_4=\rho''/\rho'$, $\delta_3=\mu'/\mu''$, $\delta_1=\delta_3\delta_4$, $\delta(\ )$ is the dirac delta function, and

$$\xi_2=0,\ \theta_1<\xi_1<\theta_2,\ -\theta_3<\xi_1<-\theta_1, \tag{14}$$

($\theta_1$, $\theta_2$, $\theta_3$ initially unknown) represents the noncontact lower boundary, assumed for the present to be two regions. The parameter $p$ is chosen so that the origin of the coordinate systems occurs at the midpoint of the central contact region. Regularity conditions will be required to assure that the slope at the bottom surface at the contact points is continuous. Finally all physically admissible solutions must be such that the normal stress $\sigma^*_{22}(\xi_1,0)$ remain compressive in the contact region, and the normal displacement $u^*_2(\xi_1,0)$ be positive in the noncontact region.

## INFINITE STRIP SOLUTIONS

Considering the infinite strip and applying the exponential Fourier transform to (2), (4), (7)–(9), the following integral expressions (see[10] for details) are obtained for the normal displacement and normal stress on the lower boundary of the strip

$$u'_2(\xi_1,0)=\frac{i\beta_2'^2}{2\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\omega B(\omega)\,e^{-i\omega\xi_1}\,d\omega, \tag{15}$$

$$\sigma'_{22}(\xi_1,0)=\frac{i}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\omega B(\omega)R(\omega)\,e^{-i\omega\xi_1}\,d\omega-P\int_{-\infty}^{\infty}S(\omega)\,e^{-i\omega(\xi_1-p)}\,d\omega, \tag{16}$$

where

$$\begin{gathered}S(\omega)=(\alpha\sinh\lambda_2-\sinh\lambda_1)/2\pi\Delta,\ \Delta=\alpha\cosh\lambda_1\sinh\lambda_2-\cosh\lambda_2\sinh\lambda_1,\\ \alpha=\left(1-\frac{1}{2}\beta_2'^2\right)\Big/\kappa_1'\kappa_2',\ \lambda_\gamma=\omega\kappa_\gamma',\ \kappa_\gamma'=(1-\beta_\gamma'^2)^{1/2},\ \gamma=1,2\\ R(\omega)=[2\lambda_2/\Delta][2\alpha(1-\cosh\lambda_1\cosh\lambda_2)+(\alpha^2+1)\sinh\lambda_1\sinh\lambda_2],\end{gathered} \tag{17}$$

and $B(\omega)$ is unknown. Now (15) and (16) will be decomposed into even and odd functions of $\xi_1$. This is accomplished by writing

$$\omega B(\omega)=B_1(\omega)+B_2(\omega),$$

where

$$B_1(\omega)=-B_1(\omega),\ B_2(\omega)=B_2(-\omega), \tag{18}$$

and recognizing that $R(\omega)$ and $S(\omega)$ are even functions of $\omega$, which gives

$$u_2'(\xi_1, 0) = \frac{\beta_2'^2}{\sqrt{(2\pi)}} \int_0^\infty [iB_2(\omega)\cos\omega\xi_1 + B_1(\omega)\sin\omega\xi_1]\,\mathrm{d}\omega, \tag{19}$$

$$\sigma_{22}'(\xi_1, 0) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty [iB_2(\omega)\cos\omega\xi_1 + B_1(\omega)\sin\omega\xi_1]R(\omega)\,\mathrm{d}\omega. \tag{20}$$

## HALF PLANE SOLUTION

Applying the exponential Fourier transform to (2), (4), (6) and (10), the normal displacement and normal stress on the boundary of the half plane become

$$u_2''(\xi_1, 0) = \frac{i\beta_2''^2}{2\sqrt{(2\pi)}\kappa_2''} \int_{-\infty}^\infty \operatorname{sgn}(\omega) C(\omega)\,\mathrm{e}^{-i\omega\xi_1}\,\mathrm{d}\omega, \tag{21}$$

$$\sigma_{22}''(\xi_1, 0) = \frac{iQ}{\sqrt{(2\pi)}} \int_{-\infty}^\infty \omega C(\omega)\,\mathrm{e}^{-i\omega\xi_1}\,\mathrm{d}\omega, \tag{22}$$

where

$$Q = [4\kappa_1''\kappa_2'' - (2 - \beta_2''^2)^2]/2\kappa_1''\kappa_2'', \tag{23}$$

and $C(\omega)$ is unknown.

The quantity $Q$ defined in (23) and $\alpha$ in (17) are closely related to the Rayleigh functions of the half plane and strip respectivley. Decomposing $C(\omega)$ according to

$$C(\omega) = C_1(\omega) + C_2(\omega)$$

where

$$C_1(\omega) = -C_1(-\omega),\ C_2(\omega) = C_2(-\omega), \tag{24}$$

eqns (21), (22) become

$$u_2''(\xi_1, 0) = \frac{\beta_2''^2}{\sqrt{(2\pi)}\kappa_2''} \int_0^\infty [iC_2(\omega)\cos\omega\xi_1 + C_1(\omega)\sin\omega\xi_1]\,\mathrm{d}\omega, \tag{25}$$

$$\sigma_{22}''(\xi_1, 0) = \sqrt{\left(\frac{2}{\pi}\right)}\, Q \int_0^\infty \omega[iC_2(\omega)\cos\omega\xi_1 + C_1(\omega)\sin\omega\xi_1]\,\mathrm{d}\omega. \tag{26}$$

## CONTINUITY AT THE INTERFACE

By applying continuity of normal stress (11) the unknowns $C_2(\omega)$ and $C_1(\omega)$ are expressed directly in terms of $B_2(\omega)$ and $B_1(\omega)$ using (20) and (26) and appear as

$$Q\delta_4\omega C_2(\omega) = i\sqrt{(2\pi)}PS(\omega)\cos\omega p + B_2(\omega)R(\omega), \tag{27}$$

$$Q\delta_4\omega C_1(\omega) = -\sqrt{(2\pi)}PS(\omega)\sin\omega p + B_1(\omega)R(\omega). \tag{28}$$

For convenience we define $\Omega$ as the noncontact region (14), and $R$ as the real line $\xi_2 = 0$, $-\infty < \xi_1 < \infty$. The first mixed condition (12) can be satisfied by the following integral representations

$$B_2(\omega) - \frac{\delta_1^2}{\kappa_2''} C_2(\omega) = -\frac{i\sqrt{(2\pi)}}{\beta_2'^2\omega} \int_\Omega \phi(t)\sin\omega t\,\mathrm{d}t, \tag{29}$$

$$B_1(\omega) - \frac{\delta_1^2}{\kappa_2''} C_1(\omega) = -\frac{\sqrt{(2\pi)}}{\beta_2'^2\omega} \int_\Omega \phi(t)\cos\omega t\,\mathrm{d}t, \tag{30}$$

along with (19) and (25) obtaining

$$u_2'(\xi_1, 0) - \delta_1 u_2''(\xi_1, 0) = \frac{\pi}{2}\int_\Omega \phi(t)[H(|t| - |\xi_1|)\,\mathrm{sgn}\,(t) - H(|\xi_1| - |t|)\,\mathrm{sgn}\,(\xi_1)]\,\mathrm{d}t, \tag{31}$$

in which the identity [14; p. 18, eqn 1]

$$\int_0^\infty p^{-1}\sin px \cos py\,\mathrm{d}p = (\pi/2)\,\mathrm{sgn}\,(x)H(|x| - |y|)$$

was used, where $H(x)$ is the unit step function. Now using $H(x) = 1 - H(-x)$ in (29) and imposing

$$\int_{\theta_1}^{\theta_2}\phi_2(t)\,\mathrm{d}t = 0,\ \int_{-\theta_3}^{-\theta_1}\phi_1(t)\,\mathrm{d}t = 0 \tag{32}$$

where

$$\phi(t) = \begin{cases}\phi_1(t), & -\theta_3 < t < -\theta_1,\\ \phi_2(t), & \theta_1 < t < \theta_2,\end{cases} \tag{33}$$

the relative normal iterfacial displacement finally becomes

$$u_2'(\xi_1, 0) - \delta_1 u_2''(\xi_1, 0) = \begin{cases}\pi\int_\Omega \phi(t)H(t - \xi_1)\,\mathrm{d}t, & \xi_1 \epsilon \Omega\\ 0, & \xi_1 \epsilon R - \Omega\end{cases} \tag{34}$$

which satisfies (12).

In order to satisfy (13) the normal stress is written as

$$\sigma_{22}'(\xi_1, 0) = (2/\beta_2'^2)\int_\Omega \phi(t)\int_0^\infty \omega^{-1}\bar{R}(\omega)\sin\omega(t - \xi_1)\,\mathrm{d}\omega\,\mathrm{d}t - 2P\bar{\gamma}(\xi_1 - p) \tag{35}$$

where

$$\bar{R}(\omega) = \omega R(\omega)/[\omega - GR(\omega)],\ G = \delta_1\delta_3/Q\kappa_2'',$$
$$\bar{\gamma}(x) = \int_0^\infty \bar{S}(\omega)\cos\omega x\,\mathrm{d}\omega,\ \bar{S}(\omega) = \omega S(\omega)/[\omega - GS(\omega)]. \tag{36}$$

In obtaining (35), the relations (27)–(30) were substituted into (20). Now decomposing $\bar{R}(\omega)$ according to

$$\begin{aligned}&\bar{R}(\omega) = -\alpha_2\omega[1 - \bar{k}(\omega)],\\ &\bar{k}(\omega) = k(\omega)/\{1 + 2\kappa_2'(1 - \alpha)G[1 - k(\omega)]\},\\ &k(\omega) = -\{\alpha^2 e^{-\lambda_1}\sinh\lambda_2 + \alpha[e^{-\lambda_1}\cosh\lambda_2 + e^{-\lambda_2}\cosh\lambda_1 - 2] + e^{-\lambda_2}\sinh\lambda_1\}/(1 - \alpha)\Delta(\omega)\end{aligned} \tag{37}$$

and using

$$\frac{\mathrm{d}}{\mathrm{d}y}\int_0^\infty p^{-1}\sin px \sin py\,\mathrm{d}p = \frac{x}{x^2 - y^2},$$

which can be obtained by differentiating [14; p. 78, eqn 1], the foundation contact pressure becomes

$$r^*(\xi_1) = 1 + 2\alpha_3\int_\Omega \frac{\phi(t)}{t - \xi_1}\,\mathrm{d}t + 2\alpha_3\int_\Omega \phi(t)\bar{K}(\xi_1 - t)\,\mathrm{d}t + 2P\bar{\gamma}(\xi_1 - p) \tag{38}$$

where

$$\bar{K}(x) = \int_0^\infty \bar{k}(\omega) \sin \omega x \, d\omega, \; \alpha_3 = \alpha_2/\beta_2'^2. \tag{39}$$

SOLUTION TYPES

For sufficiently small combinations of force and speed it can be expected that the layer would be in complete contact with the foundation. The possibility of such solutions existing is investigated by setting $\Omega = 0$, $p = 0$ in (38) obtaining

$$r^*(\xi_1) = 1 + 2P\bar{\gamma}(\xi_1). \tag{40}$$

This expression is then evaluated numerically subject to the condition that $r^*(\xi_1) > 0$ for $-\infty < \xi_1 < \infty$. The maximum force $P$ for complete contact is shown in Fig. 2 for a range of speed and different material properties. Separation initially occurs at two points located symmetrically about the applied load, as indicated in Fig. 3.

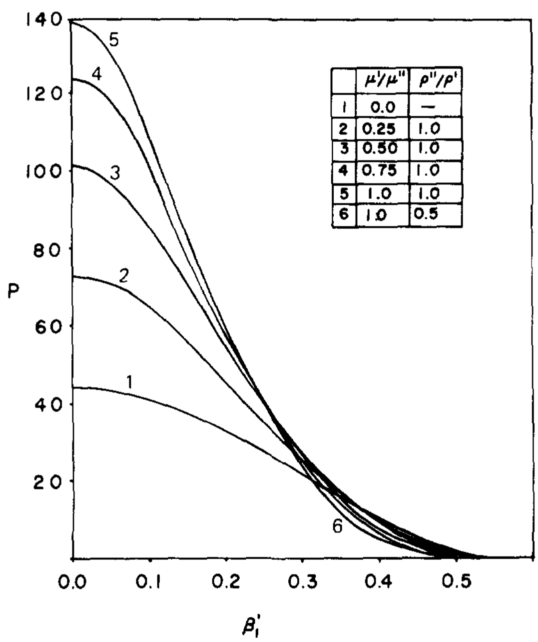


Fig. 2. Force '$P$' required to initiate separation vs speed $\beta_1'$.

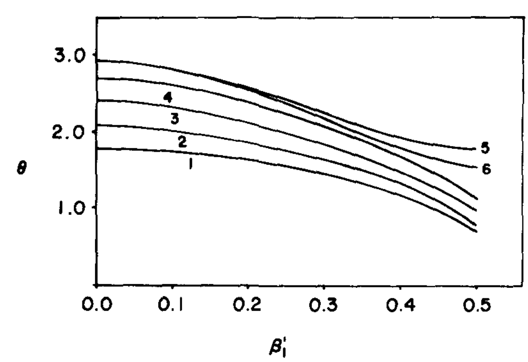


Fig. 3. Location of points of initial separation $\theta$ vs speed $\beta_1'$.

The possible existence of two noncontact regions is investigated by setting $\Omega = (-\theta_3, -\theta_1)\nu(\theta_1, \theta_2)$, using (33), (38) and (13) obtaining

$$\alpha_3 \int_{-\theta_3}^{-\theta_1} \left[\frac{1}{t-\xi_1} + \bar{K}(\xi_1 - t)\right]\phi_1(t)\,\mathrm{d}t + \alpha_3 \int_{\theta_1}^{\theta_2} \left[\frac{1}{t-\xi_1} + \bar{K}(\xi_1 - t)\right]\phi_2(t)\,\mathrm{d}t + P\bar{\gamma}(\xi_1 - p) = -1/2,$$
$$-\theta_3 < \xi_1 < -\theta_1,\ \theta_1 < \xi_1 < \theta_2, \qquad (41)$$

subject to (32). This system of singular integral equations will be solved numerically with the collocation method of Erdogan and Gupta[13]. An iterative scheme will be necessary in order to determine the additional unknowns of $\theta_1$, $\theta_2$, $\theta_3$ and $p$. The foundation contact pressure anywhere in the contact region can then be determined from (38). Since the index of the equation is $-1$, consistency conditions are also necessary[15]. These conditions will be incorporated directly with the numerical scheme of[13]. The regularity conditions of continuous slope are then automatically satisfied.

Solutions symmetric about the applied load may also be obtained from (38) by setting $p = 0$, $\theta_3 = \theta_2$, and $r^*(\xi_1) = r^*(-\xi_1)$. This results in $\phi_1(-t) = -\phi_2(t)$ and hence (38) becomes

$$r^*(\xi_1) = 1 + 4\alpha_3 \int_{\theta_1}^{\theta_2} \frac{t\phi_2(t)}{t^2 - \xi_1^2}\,\mathrm{d}t + 2\alpha_3 \int_{\theta_1}^{\theta_2} [\bar{K}(\xi_1 - t) - \bar{K}(\xi_1 + t)]\phi_2(t)\,\mathrm{d}t + 2P\bar{\gamma}(\xi_1 - p),\ 0 < \xi_1 < \infty. \qquad (42)$$

Then setting

$$r^*(\xi_1) = 0,\ \theta_1 < \xi_1 < \theta_2 \qquad (43)$$

a singular integral equation for $\phi_2(t)$ is obtained subject to (32).

The existence of a single noncontact region is determined by arbitrarily setting $\theta_1 = \theta_3 = 0$ in (38). Setting

$$r^*(\xi_1) = 0,\ 0 < \xi_1 < \theta_2 \qquad (44)$$

yields a singular integral equation subject to $(32)_1$. Symmetric solutions of this type may be obtained from (41) with the range of integration as well as the limits evaluated from $-\theta_2$ to $\theta_2$, with $p = 0$.

Solutions for a stationary load can be obtained by taking an asymptotic expansion for small speeds or by resolving the problem. For brevity we simply list the results, which are always symmetric and of the same from as (42) and (43), with

$$\begin{aligned}
&\bar{K}(x) = \int_0^\infty \bar{k}(\omega)\sin \omega x\,\mathrm{d}\omega,\ \bar{\gamma}(x) = \int_0^\infty \bar{S}(\omega)\cos \omega x\,\mathrm{d}\omega,\\
&\bar{k}(\omega) = (1-\nu')k(\omega)/h(\omega),\ \bar{S}(\omega) = (1-\nu')S(\omega)/h(\omega),\\
&h(\omega) = 1 - \nu' + \delta_3(1-\nu'')[1 - k(\omega)],\ \Delta(\omega) = \sinh 2\omega + 2\omega,\\
&k(\omega) = [2\omega(\omega+1) + 1 - \mathrm{e}^{-2\omega}]/\Delta(\omega),\\
&S(\omega) = (\omega \cosh \omega + \sinh \omega)/\Delta(\omega),\ \alpha_3 = 1/2[(1-\nu'')\delta_3 + (1-\nu')].
\end{aligned} \qquad (45)$$

## NUMERICAL SOLUTIONS

The singular integral equations previously defined will be solved using the collocation method of Erdogan and Gupta[13]. Attention will be focused on the solution of the system of eqns (41) and (32) which corresponds to the non-symmetric case with two regions of noncontact. It is first necessary to normalize the system of equations by the following linear transformations

$$s_i = \frac{2t}{(b_i - a_i)} - \frac{(b_i + a_i)}{(b_i - a_i)},\ r_i = \frac{2\xi_1}{(d_i - c_i)} - \frac{(d_i + c_i)}{(d_i - c_i)},\ i = 1,2 \qquad (46)$$

where

$$\begin{aligned} a_1 &= \theta_1,\ b_1 = \theta_2, \qquad \theta_1 < t < \theta_2 \\ a_2 &= -\theta_3,\ b_2 = -\theta_1,\ -\theta_3 < t < -\theta_1 \\ c_1 &= \theta_1,\ d_1 = \theta_2, \qquad \theta_1 < \xi <_1 < \theta_2 \\ c_2 &= -\theta_3,\ d_2 = -\theta_1,\ -\theta_3 < \xi_1 < -\theta_1. \end{aligned}$$

Then defining

$$\begin{aligned} \Psi_i(s_i) &= \phi_i(t),\ i = 1,2 \\ \bar{\Psi}_i(s_i) &= (1 - s_i^2)^{-1/2}\Psi_i(s_i),\ i = 1,2 \end{aligned} \tag{47}$$

eqns (32) and (41) are approximated (using the method of[10]) by the following system of $2N + 4$ linear algebraic equations

$$\begin{aligned} &\frac{\alpha_3 \pi}{N+1} \sum_{i=1}^{2} \sum_{J=1}^{N} (1 - s_{iJ}^2)\bar{\Psi}_i(s_{iJ}) \left[ \frac{1}{t_{iJ} - \xi_{1iK}} + K(\xi_{1ik} - t_{iJ}) \right] l_i \\ &+ P\gamma(\xi_{1iK} - p) = -1/2,\ K = 1,2,\ldots,N+1,\ i = 1,2 \\ &\sum_{J=1}^{N} (1 - s_{iJ}^2)\bar{\Psi}_i(s_{iJ}) = 0,\ i = 1,2 \end{aligned} \tag{48}$$

where[13],

$$\begin{aligned} s_{iJ} &= \cos\left(\frac{J\pi}{N+1}\right),\ J = 1,2,\ldots,N,\ i = 1,2 \\ r_{iK} &= \cos\left(\frac{\pi(2K-1)}{2N+2}\right),\ K = 1,2,\ldots,N+1,\ i = 1,2 \end{aligned}$$

and from (46)

$$\begin{aligned} t_{iJ} &= \frac{b_i - a_i}{2} s_{iJ} + \frac{b_i + a_i}{2},\ J = 1,2,\ldots,N,\ i = 1,2 \\ \xi_{1iK} &= \frac{d_i - c_i}{2} r_{iK} + \frac{d_i + c_i}{2},\ K = 1,2,\ldots,N+1,\ i = 1,2 \\ l_i &= \frac{b_i - a_i}{2},\ i = 1,2. \end{aligned}$$

The system of eqns (48) is linear in the $2N$ unknowns $\bar{\Psi}_i(s_{iJ})$ but non-linear in $\theta_1$, $\theta_2$, $\theta_3$ and $p$. However, in order to facilitate computations $\theta_3$ may be considered known and $P$ unknown. Hence the system is linear in $2N + 1$ unknowns and an iterative technique is necessary to solve for $\theta_1$, $\theta_2$ and $p$. It is noted that the parameter $p$ enters into the equations only through $(36)_3$ and for specified $\theta_1$, $\theta_2$ can be determined very efficiently. By slowly changing $\theta_3$, good initial guesses could be made for $\theta_1$ and $\theta_2$. Having thus determined $\theta_1$, $\theta_2$, $p$ and the function $\Psi(t)$, the contact pressure in the contact region may be determined by a quadrature of (38), and the relative normal displacement in the noncontact region from (32) to (34).

## RESULTS AND DISCUSSION

In Fig. 4 is a plot of force $P$ vs noncontact region location as defined by $\theta_1$, $\theta_2$, $\theta_3$ for fixed speed ($\beta_1' = 0.4$) and material properties ($\mu'/\mu'' = 0.5$, $\rho''/\rho' = 1.0$). The related lower boundary displacements and foundation contact pressures are shown in Fig. 5(a–f). These configurations correspond to symmetric solutions (a, d), nonsymmetric (c, f), and nonsymmetric with one noncontact region (b, e). The corresponding values of $p$ are not shown but were in the range $-\theta_1 < p < 0$ for (c, f) and $\theta_2 < p$ for (b, e). It is emphasized that only the first two solutions of

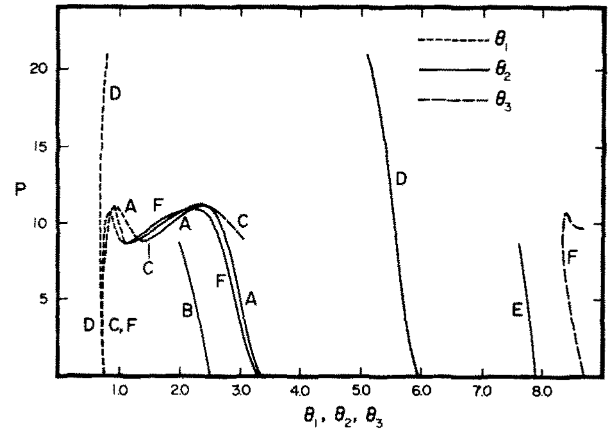


Fig. 4. Force $P$ vs non-contact region location for fixed speed ($\beta_1' = 0.4$) and material properties ($\mu'/\mu'' = 0.5$, $\rho''/\rho' = 1.0$).

each type are shown; many others exist. Nonsymmetric solution c, with two noncontact regions, gradually becomes symmetric as the applied force $P$ is increased to the maximum value possible for configuration c (Fig. 4). At that point it actually coincides with the locally maximum value for symmetric shape a. Nonsymmetric configurations (b, e) each have a single noncontact region. As the load $P$ is increased another noncontact region appears which leads to the development of nonsymmetric shapes (c, f) each of which have two noncontact regions. Solutions for more than two noncontact regions can be obtained from (38) but are not shown here.

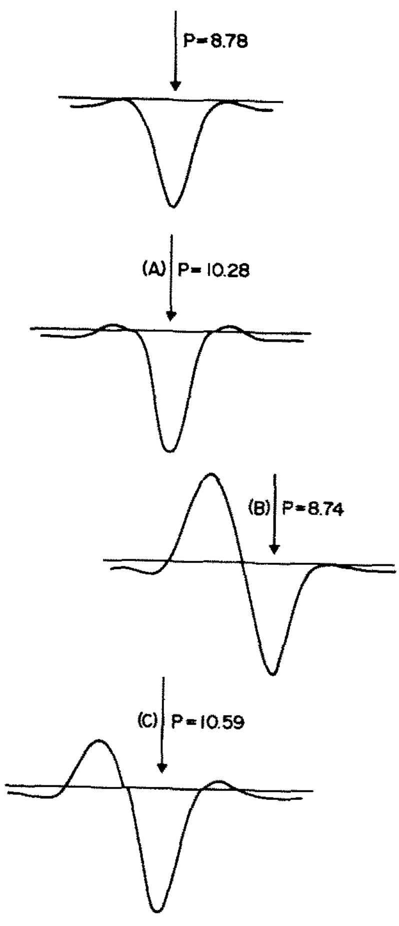


Fig. 5. Part 1.

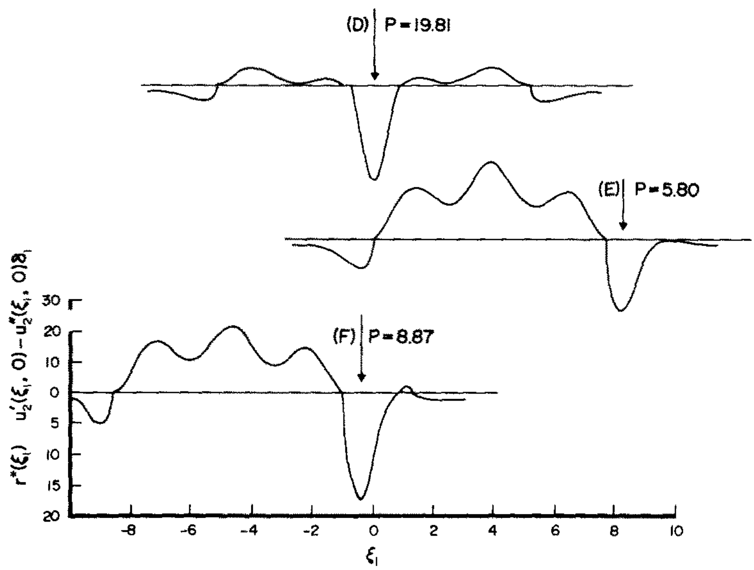


Fig. 5. Part 2.

Fig. 5. Lower boundary displacements and foundation contact pressure at fixed speed ($\beta_1' = 0.4$) and material properties ($\mu'/\mu'' = 0.5$, $\rho''/\rho' = 1.0$).

It is observed that because of the way in which the speed $c$ enters into the problem, if a certain configuration is a solution, then so is its reflection through $\xi_1 = 0$. Hence only the nonsymmetric solutions for which $\theta_3 > \theta_2$ have been shown.

In Figs. 6–8 are graphs of force $P$, foundation pressure under the load $r^*(0)$, and maximum relative normal displacement, vs noncontact lengths for fixed speed $\beta_1' = 0.4$ and different material pairs, all for symmetric configurations of the type $A$. The results of Fig. 6 show that as the stiffness of the foundation increases relative to that of the strip, the maximum force $P$ for which solutions of the type $A$ exist also increases (at this speed). From Fig. 7 the pressure beneath the load transmitted to the foundation is shown to be a maximum for the same value of $\theta_2$ for which $P$ is a maximum (Fig. 6). The relative normal displacement increases monotically with $\theta_2$ (Fig. 8). The maximum value does not occur when $P$ is a maximum. Figures 9–11 show the corresponding results for the static problem. Note that the force $P$, the pressure transmitted to the foundation, and the maximum relative displacement all increase monotically with $\theta_2$. As the foundation stiffness increases, the minimum load required for development of noncontact regions decreases. This is the reverse of the results shown in Fig. 5 but is expected due to the

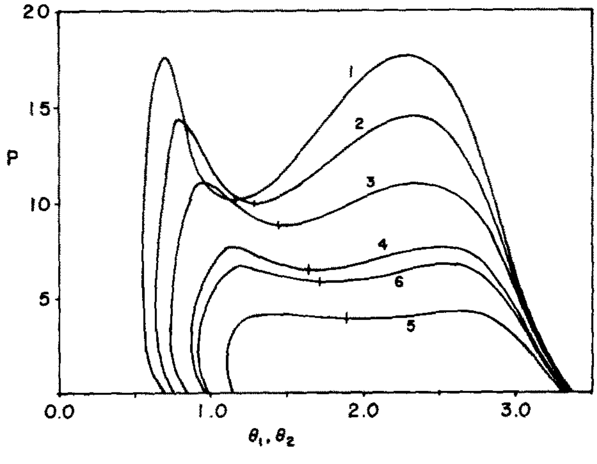


Fig. 6. Force '$P$' vs non-contact lengths $\theta_1$ and $\theta_2$, at fixed speed $\beta_1' = 0.4$.

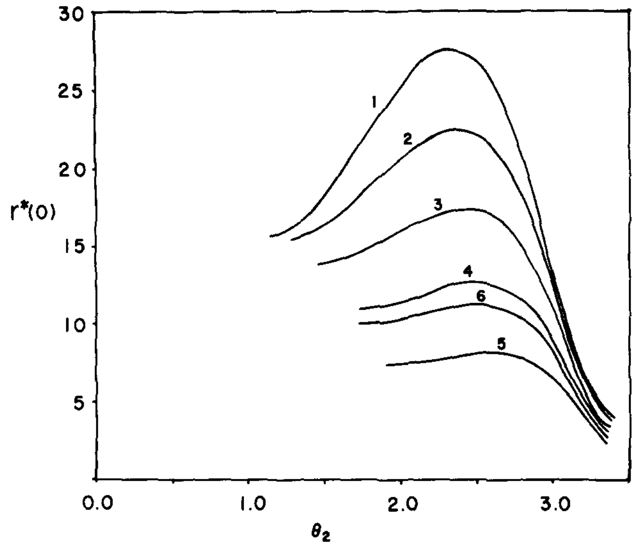


Fig. 7. Foundation contact pressure under the load $r^*(0)$ vs $\theta_2$, at fixed speed $\beta_1' = 0.4$.

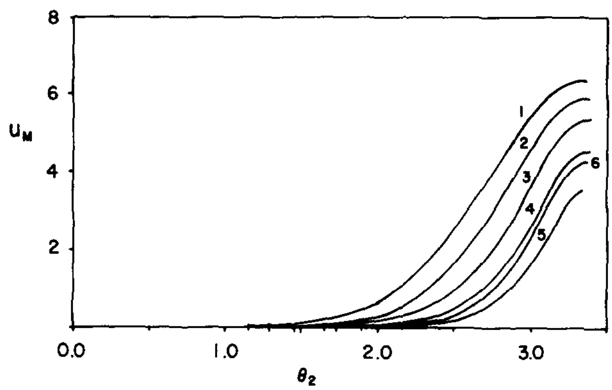


Fig. 8. Maximum relative normal displacement $u_M$ vs $\theta_2$, at fixed speed $\beta_1' = 0.4$.

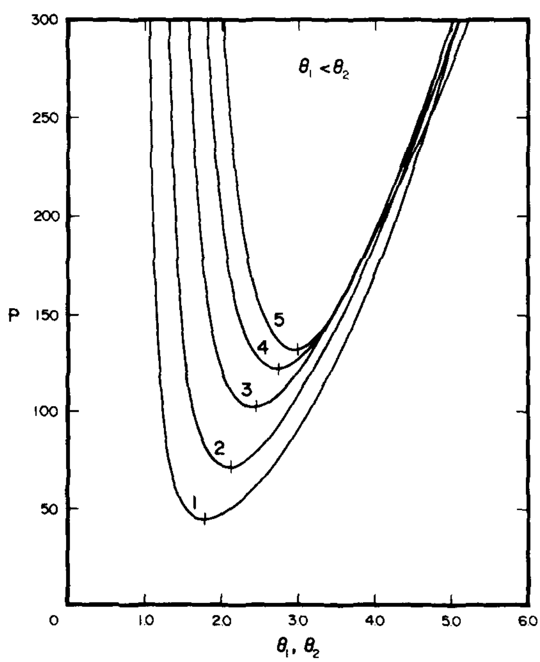


Fig. 9. Force '$P$' vs non-contact lengths $\theta_1$ and $\theta_2$ for static case.

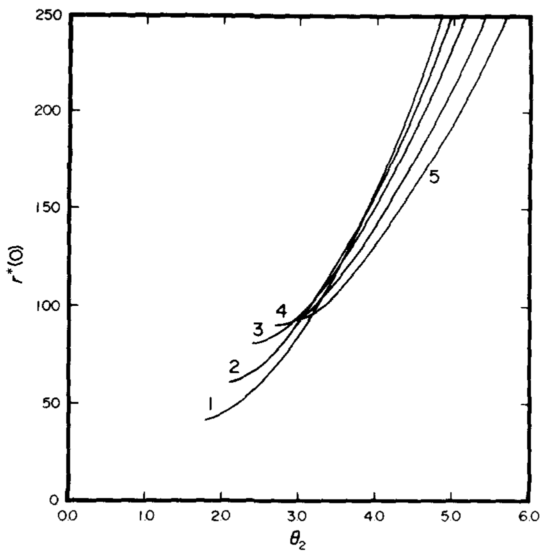


Fig. 10. Foundation contact pressure under the load $r^*(0)$ vs $\theta_2$ for static case.

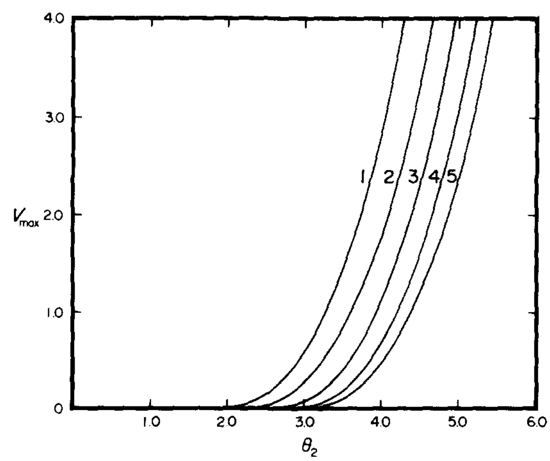


Fig. 11. Maximum relative normal displacement $v_{max}$ vs $\theta_2$ for static case.

crossing of the curves in Fig. 2. Also, in the limit for large $P$, the results obtained here agree with those of [12] and [1] in which gravity has been neglected.

The nonuniqueness of the solutions obtained is due to the steady state solution of this contact problem. Although uniqueness theorems of elastodynamics apply only to initial value problems, one may expect unique solutions for the steady problem (except perhaps at certain critical speeds) in cases where the principle of superposition holds. Such is not the case here due to the non-linearity associated with the existance of noncontact regions. All solutions given here satisfy the equations of motion, boundary conditions, have compressive contact stresses, and do not violate material interference conditions. However, some may not produce local minimums of the total energy and, therefore, be physically unrealistic. The actual solution (which may depend upon the initial conditions) would have to be determined from the limit of the appropriate initial value problem.

## REFERENCES

1. L. M. Keer, J. Dundurs and K. C. Tsai, Problems involving a receding contact between a layer and a half space. *J. Appl. Mech.* **39**, *Trans. ASME* **94E**, 1115–1120 (1972).
2. K. C. Tsai, J. Dundurs and L. M. Keer, Elastic layer pressed against a half space. *ASME J. Appl. Mech.* **41**, 703–707 (1974).
3. K. C. Tsai, J. Dundurs and L. M. Keer, Contact between an elastic layer with a slightly curved bottom and a substrate. *ASME J. Appl. Mech.* **39**, 821–823 (1972).

4. F. Erdogan and M. Ratwani, On the plane contact problem of a frictionless elastic layer. *Int. J. Solids Structures* **9**, 921–936 (1973).
5. M. B. Civelek and F. Erdogan, The axisymmetric double contact problem for a frictionless elastic layer. *Int. J. Solids Structures* **10**, 639–659 (1974).
6. L. M. Keer and M. A. G. Silva, Two mixed problems of a semi-infinite layer. *J. Appl. Mech.* **39**, *Trans. ASME*, **94E**, 1121–1124 (1972).
7. M. B. Civelek and F. Erdogan, The frictionless contact problem of an elastic layer under gravity. *J. Appl. Mech.* **42**, *Trans. ASME* **97E**, 136–140 (1975).
8. M. B. Civelek and F. Erdogan, Interface separation in a frictionless contact problem for an elastic layer. *J. Appl. Mech.* **43**, *Trans. ASME* **98E**, 175–177 (1976).
9. G. G. Adams and D. B. Bogy, Steady solutions for moving loads on elastic beams with one-sided constraints. *J. Appl. Mech.* **42**, *Trans. ASME* **97E**, 800–804 (1975).
10. G. G. Adams, Moving loads on elastic strips with one-sided constraints. *Int. J. Engng Sci.* **14**, 1071–1083 (1976).
11. G. G. Adams, A steadily moving load on an elastic strip resting on a rigid foundation. *Int. J. Engng Sci.* **16**, 659–667 (1978).
12. G. G. Adams, An elastic strip pressed against an elastic half plane by a steadily moving force. *J. Appl. Mech.* **45**, *Trans. ASME* **100E**, 89–94 (1978).
13. F. Erdogan and G. D. Gupta, On the numerical solution of singular integral equations. *Q. J. Appl. Math.* **29**, 525–534 (1972).
14. A. Erdelyi *et al.*, *Tables of Integral Transforms*, Vol. I, p. 18. McGraw-Hill, New York (1954).
15. N. I. Muskhelishvili, *Singular Integral Equations*. Norkhoff, Groningen (1953).